Cotorsioness of modules and a problem of George Bergman

ABSTRACT

Let I be an infinite set and let $\{A_i : i \in I\}$ be a collection of non-trivial groups (or a collection of non-trivial rings, or ....modules or ....lattices or .... monoids). Their direct product $P = \prod_{i \in I} A_i$ is always big − the size of P is $\geq 2^{\aleph_0}$. But there are objects A (groups, rings, modules, or monoids) having the property that for every homomorphism $f : P \to A$, the image $f(P)$ is "small" in the sense that $f(A_i) = 0$ for almost all $i \in I$. In a recent paper (Pacific J. Math, vol. 274 (2015)), George Bergman investigated conditions under which every homomorphism $f : P \to A$ vanishes on some ultra product of the A_i. In that context he was lead to the following simple problem which he left open.

Problem: Let $P = \prod_{i \in I} A_i$ and let $S = \bigoplus_{i \in I} A_i$ be the direct sum of the A_i. Let A be an object of the same kind as the A_i (a group or a ring or module or lattice or monoid). Characterize A if every homomorphism from S to A extends to a homomorphism from P to A.

If the A_i are modules over a Dedekind Domain and, in particular, are abelian groups, then I will show that the modules A with the stated property are precisely a well-known type of modules called cotorsion modules. We will discuss the various properties of P, its homomorphic images, the class of cotorsion modules etc and use them to answer the question of Bergman. In addition to discussing the related open problems, we shall also discuss the generalization to cotorsion theories for modules over integral domains involving the three types of cotorsioness − the Enochs cotorsion, the Matlis cotorsion and the Warfield cotorsion modules.